

On uniformly recurrent subgroups of finitely generated groups*

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Abstract

We prove that if G is a finitely generated group and Z is a uniformly recurrent subgroup of G then there exists a minimal system (X, G) with Z as its stability system. This answers a query of Glasner and Weiss [7] in the case of finitely generated groups. Using the same method (introduced by Alon, Grytczuk, Haluszczak and Riordan [2]) we will prove that finitely generated sofic groups have free Bernoulli-subshifts admitting an invariant probability measure.

Keywords. uniformly recurrent subgroups, sofic groups

1 Introduction

Let Γ be a countable group and $\text{Sub}(\Gamma)$ be the compact space of all subgroups of Γ . The group Γ acts on $\text{Sub}(\Gamma)$ by conjugation. *Uniformly recurrent subgroups* (URS) were defined by Glasner and Weiss [7] as closed, invariant subsets $Z \subset \text{Sub}(\Gamma)$ such that the action of Γ on Z is minimal (every orbits are dense). Now let (X, Γ, α) be a Γ -system (that is, X is a compact metric space and $\Gamma \rightarrow \text{Homeo}(X)$ is a homomorphism). For each point $x \in X$ one can define the topological stabilizer subgroup $\text{Stab}_\alpha^0(x)$ by

$$\text{Stab}_\alpha^0(x) = \{\gamma \in \Gamma \mid \gamma \text{ fixes some neighborhood of } x\}.$$

Let us consider the Γ -invariant subset $X^0 \subseteq X$ such that $x \in X^0$ if and only if $\text{Stab}_\alpha(x) = \text{Stab}_\alpha^0(x)$. Then X^0 is a dense G_δ -set and we have a Γ -equivariant map $S_\alpha : X^0 \rightarrow \text{Sub}(\Gamma)$ such that if $y \in X^0$ then $S_\alpha(y) = \text{Stab}_\alpha(y)$. The closure of the invariant subset $S_\alpha(X^0) \subset \text{Sub}(\Gamma)$ is called the *stability system* of (X, Γ, α) (see also [12], [10]). If the action is minimal, then the stability system of (X, Γ, α) is an URS. Glasner and Weiss proved (Proposition 6.1, [7]) that for every URS $Z \subset \text{Sub}(G)$ there exists a topologically transitive (that is there is a dense orbit) system (X, Γ, α) with Z as its stability system. They asked (Problem

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6.2., [7]), whether for any URS Z there exists a minimal system (X, Γ, α) with Z as its stability system. Recently, Kawabe [12] gave an affirmative answer for this question in the case of amenable groups. We will prove the following result.

Theorem 1. *If Γ is a finitely generated group and $Z \subset \text{Sub}(\Gamma)$ is an URS, then there exists a minimal system (X, Γ, α) with Z as its stability system.*

In the proof we will use the Lovász Local Lemma technique of Alon, Grytczuk, Haluszczak and Riordan [2] to construct a minimal action on the space of rooted colored Γ -Schreier graphs. This approach has already been used to construct free Γ -Bernoulli subshifts by Aubrun, Barbieri and Thomassé [1]. The other result of the paper is about free Γ -Bernoulli-subshifts, that is closed Γ -invariant subsets M of K^Γ , where K is some finite alphabet and the action of Γ on M is free. For a long time all finitely generated groups that had been known to have free Bernoulli-subshifts were residually-finite. Then Dranishnikov and Schroeder [11] constructed a free Bernoulli-subshift for any torsion-free hyperbolic group. Somewhat later Gao, Jackson and Seward proved that any countable group has free Bernoulli-subshifts [5], [6]. On the other hand, Hjorth and Molberg [9] proved that for any countable group Γ there exists a free continuous action of Γ on a Cantor set admitting an invariant measure. It seems that so far all groups Γ for which free Bernoulli-shifts with an invariant probability measure proved to exist were either residually-finite (Toeplitz-subshifts) or amenable (when the existence of invariant measure is obvious). Using the coloring technique of Alon, Grytczuk, Haluszczak and Riordan we prove a combination of these results for finitely generated sofic groups.

Theorem 2. *Let Γ be a finitely generated sofic group. Then there exists a free Bernoulli-subshift for Γ .*

2 The space of colored rooted Γ -Schreier graphs

Let Γ be a finitely generated group with a minimal symmetric generating system $Q = \{\gamma_i\}_{i=1}^n$. Let $H \in \text{Sub}(\Gamma)$. Then the Schreier graph of H , $S_\Gamma^Q(H)$ is constructed as follows.

- The vertex set of $S_\Gamma^Q(H)$, $V(S_\Gamma^Q(H)) = \Gamma/H$ (that is Γ acts on the vertex set of $S_\Gamma^Q(H)$ on the left).
- The vertices corresponding to the cosets aH and bH are connected by a directed edge labeled by the generator γ_i if $\gamma_i aH = bH$.

The coset class of H is called the *root* of the graph $S_\Gamma^Q(H)$. We will consider the usual shortest path distance on $S_\Gamma^Q(H)$ and denote the ball of radius r around the root H by $B_r(S_\Gamma^Q(H), H)$. Note that $B_r(S_\Gamma^Q(H), H)$ is a rooted edge-labeled graph. The space of all Schreier graphs S_Γ^Q is a compact metric space, where

$$d_{S_\Gamma^Q}(S_\Gamma^Q(H_1), S_\Gamma^Q(H_2)) = 2^{-r},$$

if r is the largest integer for which the r -balls $B_r(S_\Gamma^Q(H_1), H_1)$ and $B_r(S_\Gamma^Q(H_2), H_2)$ are rooted-labeled isomorphic. Clearly, $s : \text{Sub}(\Gamma) \rightarrow S_\Gamma^Q$, $s(H) = S_\Gamma^Q(H)$ is a homeomorphism commuting with the Γ -actions. Note that if $\gamma \in \Gamma$ and $H \in \text{Sub}(\Gamma)$, then

$$\gamma(S_\Gamma^Q(H)) = S_\Gamma^Q(\gamma H \gamma^{-1}),$$

where the underlying labeled graphs of $S_\Gamma^Q(H)$ and $S_\Gamma^Q(\gamma H \gamma^{-1})$ are isomorphic. The graph $S_\Gamma^Q(\gamma H \gamma^{-1})$ can be regarded as the same graph as $S_\Gamma^Q(H)$ with the new root $\gamma(\text{root}(S_\Gamma^Q(H)))$. We will use the root-change picture of the Γ -action on S_Γ^Q later in the paper.

Now let K be a finite alphabet. A rooted K -colored Schreier graph is a rooted Schreier graph S_H^Q equipped with a vertex-coloring $c : \Gamma/H \rightarrow K$. Let $S_\Gamma^{K,Q}$ be the set of all rooted K -colored Schreier-graphs. Again, we have a compact, metric topology on $S_\Gamma^{K,Q}$:

$$d_{S_\Gamma^{K,Q}}(S, T) = 2^{-r},$$

if r is the largest integer such that the r -balls around the roots of the graphs S and T are rooted-colored-labeled isomorphic. We define $d_{S_\Gamma^{K,Q}}(S, T) = 2$ if the 1-balls around the roots are nonisomorphic and even the colors of the roots are different. Again, Γ acts on the compact space $S_\Gamma^{K,Q}$ by the root-changing map. Hence, we have a natural color-forgetting map $F : S_\Gamma^{K,Q} \rightarrow S_\Gamma^Q$ that commutes with the Γ -actions. Notice that if a sequence $\{S_n\}_{n=1}^\infty \subset S_\Gamma^{K,Q}$ converges to $S \in S_\Gamma^{K,Q}$, then for any $r \geq 1$ there exists some integer $N_r \geq 1$ such that if $n \geq N_r$ then the r -balls around the roots of the graph S_n and the graph S are rooted-colored-labeled isomorphic. Let $H \in \text{Sub}(\Gamma)$ and $c : \Gamma/H \rightarrow K$ be a vertex coloring that defines the element $S_{H,c} \in S_\Gamma^{K,Q}$. Then of course, $\gamma(S_{H,c}) = S_{H,c}$ if $\gamma \in H$. On the other hand, if $\gamma(S_{H,c}) = S_{H,c}$ and $\gamma \notin H$ then we have the following lemma that immediately follows from the definitions of the Γ -actions.

Lemma 2.1. *Let $\gamma \notin H$ and $\gamma(S_{H,c}) = S_{H,c}$. Then there exists a colored-labeled graph-automorphism of the K -colored labeled graph $S_{H,c}$ moving the vertex representing H to the vertex representing $\gamma(H) \neq H$.*

Note that we have a continuous Γ -equivariant map $\pi : S_\Gamma^{K,Q} \rightarrow \text{Sub}(\Gamma)$, where $\pi(t) = s^{-1} \circ F(t)$. Let Z be an URS of Γ . Let $H \in Z$ and let $t \in S_\Gamma^{K,Q}$ be corresponding to a vertex coloring of the Schreier graph $S_\Gamma^Q(H)$. We say that the element $t \in S_\Gamma^{K,Q}$ is Z -proper if $\text{Stab}_\alpha(t) = H$, where α is the right action of Γ on $S_\Gamma^{K,Q}$. Note that if $H \in Z$ and t is representing the Schreier graph S_H^Q , then by Lemma 2.1, t is Z -proper if and only if there is no non-trivial colored-labeled automorphism of t .

Proposition 2.1. *Let $Y \subset S_{\Gamma}^{K,Q}$ be a closed Γ -invariant subset consisting of Z -proper elements. Let $(M, \Gamma, \alpha) \subset (Y, G, \alpha)$ be a minimal Γ -subsystem. Then for any $m \in M$, $\text{Stab}_{\alpha}^0(m) = \text{Stab}_{\alpha}(m) \in Z$. Also, $\pi(M) = Z$.*

Proof. Let $h \in \text{Stab}_{\alpha}(m)$. Then $h \in Z$, that is, h fixes the root of m . Therefore, h fixes the root of m' provided that $d_{S_{\Gamma}^{K,Q}}(m, m')$ is small enough. Thus, $h \in \text{Stab}_{\alpha}^0(m)$. Since π is a Γ -equivariant continuous map and M is a closed Γ -invariant subset, $\pi(M) = Z$. \square

3 The proof of Theorem 1

Let Z be an URS of Γ . By Proposition 2.1, it is enough to construct a closed Γ -invariant subset $Y \subset S_{\Gamma}^{K,Q}$ for some alphabet K such that all the elements of Y are Z -proper. Let $H \in Z$ and consider the Schreier graph $S = S_{\Gamma}^Q(H)$. Following [1] and [2] we call a coloring $c : \Gamma/H \rightarrow K$ nonrepetitive if for any path $(x_1, x_2, \dots, x_{2n})$ in S there exists some $1 \leq i \leq n$ such that $c(x_i) \neq c(x_{n+i})$. We call all the other colorings repetitive.

Theorem 3. *[Theorem 1 [2]] For any $d \geq 1$ there exists a constant $C(d) > 0$ such that any graph G (finite or infinite) with vertex degree bound d has a nonrepetitive coloring with an alphabet K , provided that $|K| \geq C(d)$.*

Proof. Since the proof in [2] is about edge-colorings and the proof in [1] is in slightly different setting, for completeness we give a proof using Lovász's Local Lemma, that closely follows the proof in [2]. Now, let us state the Local Lemma.

Theorem 4 (The Local Lemma). *Let X be a finite set and Pr be a probability distribution on the subsets of X . For $1 \leq i \leq r$ let \mathcal{A}_i be a set of events, where an "event" is just a subset of X . Suppose that for all $A \in \mathcal{A}_i$, $\text{Pr}(A_i) = p_i$. Let $\mathcal{A} = \cup_{i=1}^r \mathcal{A}_i$. Suppose that there are real numbers $0 \leq a_1, a_2, \dots, a_r < 1$ and $\Delta_{ij} \geq 0$, $i, j = 1, 2, \dots, r$ such that the following conditions hold:*

- *for any event $A \in \mathcal{A}_i$ there exists a set $D_A \subset \mathcal{A}$ with $|D_A \cap \mathcal{A}_j| \leq \Delta_{ij}$ for all $1 \leq j \leq r$ such that A is independent of $\mathcal{A} \setminus (D_A \cup \{A\})$,*
- *$p_i \leq a_i \prod_{j=1}^r (1 - a_j)^{\Delta_{ij}}$ for all $1 \leq i \leq r$.*

Then $\text{Pr}(\cap_{A \in \mathcal{A}} \overline{A}) > 0$.

Let G be a finite graph with maximum degree d . It is enough to prove our theorem for finite graphs. Indeed, if G' is a connected infinite graph with vertex degree bound d , then for each ball around a given vertex p we have a nonrepetitive coloring. Picking a pointwise convergent subsequence of the colorings we obtain a nonrepetitive coloring of our infinite graph G' .

Let C be a large enough number, its exact value will be given later. Let X be the set of all random $\{1, 2, \dots, C\}$ -colorings of G . Let $r = \text{diam}(G)$ and for

$1 \leq i \leq r$ and for any path P of length $2i - 1$ let $A(P)$ be the event that P is repetitive. Set

$$\mathcal{A}_i = \{A(P) : P \text{ is a path of length } 2i - 1 \text{ in } G\}.$$

Then $p_i = C^{-i}$. The number of paths of length $2j - 1$ that intersects a given path of length $2i - 1$ is less or equal than $4ij d^{2j}$. So, we can set $\Delta_{ij} = 4ij \Delta^{2j}$. Let $a_i = \frac{1}{2d^2}$. Since $a_i \leq \frac{1}{2}$, we have that $(1 - a_i) \geq \exp(-2a_i)$. In order to be able to apply the Local Lemma, we need that for any $1 \leq i \leq r$

$$p_i \leq a_i \prod_{j=1}^r \exp(-2a_j \Delta_{ij}).$$

That is

$$C^{-i} \leq a^{-i} \prod \exp(-8ija^{-j}d^{2j}),$$

or equivalently

$$C \geq a \exp\left(8 \sum_{j=1}^r \frac{j}{2^j}\right).$$

Since the infinite series $\sum_{j=1}^{\infty} \frac{j}{2^j}$ converges to 2, we obtain that for large enough C , the conditions of the Local Lemma are satisfied independently on the size of our finite graph G . This ends the proof of Theorem 3. \square

Let $|K| = C(|Q|)$ and let $c : \Gamma/H \rightarrow K$ be a nonrepetitive K -coloring that gives rise to an element $y \in S_{\Gamma}^{K,Q}$. The following proposition finishes the proof of Theorem 1.

Proposition 3.1. *All elements of the orbit closure Y of y in $S_{\Gamma}^{K,Q}$ are Z -proper.*

Proof. Let $x \in Y$ with underlying Schreier graph H' and coloring $c' : \Gamma/H \rightarrow K$. Since Z is an URS, $H' \in Z$. Indeed, $\pi^{-1}(Z)$ is a closed Γ -invariant set and $y \in \pi^{-1}(Z)$. Clearly, $\alpha(\gamma)(x) = x$ if $\gamma \in H'$. Now suppose that $\alpha(\gamma)(x) = x$ and $\gamma \notin H'$ (that is x is not Z -proper). By Lemma 2.1, there exists a colored-labeled automorphism θ of the graph x moving $\text{root}(x)$ to $\gamma(\text{root}(x)) \neq \text{root}(x)$. Now we proceed similarly as in the proof of Lemma 2. [2] or in the proof of Theorem 2.6 [1]. Let $a \in V(x)$ be a vertex such that there is no $b \in X$ such that $\text{dist}_x(b, \theta(b)) < \text{dist}_x(a, \theta(a))$. Let $(a = a_1, a_2, \dots, a_{n+1} = \theta(a))$ be a shortest path between a and $\theta(a)$. For $1 \leq i \leq n$, let $\gamma_{k_i}(a_i) = a_{i+1}$. Then let $a_{n+2} = \gamma_{k_1}(a_{n+1}), a_{n+3} = \gamma_{k_1}(a_{n+2}), \dots, a_{2n} = \gamma_{k_n}(a_{2n-1})$. Since θ is a colored-labeled automorphism, for any $1 \leq i \leq n$

$$c(a_i) = c(a_{i+n}). \tag{1}$$

Lemma 3.1. *The walk $(a_1, a_2, \dots, a_{2n})$ is a path.*

Proof. Suppose that the walk above crosses itself, that is for some i, j , $a_j = a_{n+i}$. If $(n+1) - j \geq (n+i) - (n+1) = i - 1$, then $\text{dist}(a_2, \theta(a_2)) = \text{dist}(a_2, a_{n+2}) < \text{dist}(a, \theta(a))$. On the other hand, if $(n+1) - j \leq (n+i) - (n+1) = i - 1$, then $\text{dist}(a_n, \theta(a_n)) = \text{dist}(a_n, a_{2n-1}) < \text{dist}(a, \theta(a))$. Therefore, $(a_1, a_2, \dots, a_{2n})$ is a path. \square

By (1) and the previous lemma, the K -colored Schreier-graph x contains a repetitive path. Since x is in the orbit closure of y , this implies that y contains a repetitive path as well, in contradiction with our assumption. \square

4 Sofic groups and invariant measures

First, let us recall the notion of a finitely generated sofic group. Let Γ be a finitely generated infinite group with a minimal, symmetric generating system $Q = \{\gamma_i\}_{i=1}^r$ and a surjective homomorphism $\kappa : \mathbb{F}_n \rightarrow \Gamma$ from the free group \mathbb{F}_n with generating system $\overline{Q} = \{r_i\}_{i=1}^n$ mapping r_i to γ_i . Let Cay_Γ^Q be the Cayley graph of Γ with respect to the generating system Q , that is the Schreier graph corresponding to the subgroup $H = \{1_\Gamma\}$. Let $\{G_k\}_{k=1}^\infty$ be a sequence of finite \mathbb{F}_n -Schreier graphs. We call a vertex $p \in V(G_k)$ a (Γ, r) -vertex if there exists a rooted isomorphism

$$\Psi : B_r(G_k, p) \rightarrow B_r(\text{Cay}_\Gamma^Q, 1_\Gamma)$$

such that if e is a directed edge in the ball $B_r(G_k, p)$ labeled by r_i , then the edge $\Phi(e)$ is labeled by γ_i . We say that $\{G_k\}_{k=1}^\infty$ is a sofic approximation of Cay_Γ^Q , if for any $r \geq 1$ and a real number $\varepsilon > 0$ there exists $N_{r,\varepsilon} \geq 1$ such that if $k \geq N_{r,\varepsilon}$ then there exists a subset $V_k \subset V(G_k)$ consisting of (Γ, r) -vertices such that $|V_k| \geq (1 - \varepsilon)|V(G_k)|$. A finitely generated group Γ is called sofic if the Cayley-graphs of Γ admit sofic approximations. Sofic groups were introduced by Gromov in [8] under the name of initially subamenable groups, the word “sofic” was coined by Weiss in [13]. It is important to note that all the amenable, residually-finite and residually amenable groups are sofic, but there exist finitely generated sofic groups that are not residually amenable (see the book of Capraro and Lupini [4] on sofic groups). It is still an open question whether all groups are sofic. Now let Γ be a finitely generated sofic group with generating system $Q = \{\gamma_i\}_{i=1}^n$ and a sofic approximation $\{G_k\}_{k=1}^\infty$. Using Theorem 3, for each $k \geq 1$ let us choose a nonrepetitive coloring $c_k : V(G_k) \rightarrow K$, where $|K| \geq C(|Q|)$. We can associate a probability measure μ_k on the space of K -colored \mathbb{F}_n -Schreier graphs $S_{\mathbb{F}_n}^{\overline{Q}, K}$. Note that the origin of this construction can be traced back to the paper of Benamini and Schramm [3]. For a vertex $p \in V(G_k)$ we consider the rooted K -colored Schreier graph $(G_k^{c_k}, p)$. The measure μ_k is defined as

$$\mu_k = \frac{1}{|V(G_k)|} \sum_{p \in V(G_k)} \delta(G_k^{c_k}, p),$$

where $\delta(G_k^{c_k}, p)$ is the Dirac-measure on $S_{\mathbb{F}_n}^{\overline{Q}, K}$ concentrated on the rooted K -colored Schreier graph $(G_k^{c_k}, p)$. Clearly, μ_k is invariant under the action of \mathbb{F}_n . Since the space of \mathbb{F}_n -invariant probability measures on the compact space $S_{\mathbb{F}_n}^{\overline{Q}, K}$ is compact with respect to the weak-topology, we have a convergent subsequence $\{\mu_{n_k}\}_{k=1}^\infty$ converging weakly to some probability measure μ . Let $C_{\mathbb{F}_n}^{\overline{Q}}(N)$ be the Schreier graph corresponding to the normal subgroup $N = \text{Ker}(\kappa)$. This means that we have a natural graph isomorphism from $C_{\mathbb{F}_n}^\Gamma$ to Cay_Γ^Q that changes the labels r_i to γ_i .

Proposition 4.1. *The probability measure μ is concentrated on the \mathbb{F}_n -invariant closed set Ω of nonrepetitive K -colorings on $C_{\mathbb{F}_n}^{\overline{Q}}(N)$.*

Proof. Let $U_r \subset S_{\mathbb{F}_n}^{\overline{Q}, K}$ be the clopen set of K -colored Schreier graphs G such that the ball $B_r(G, \text{root}(G))$ is not rooted-labeled isomorphic to $B_r(C_{\mathbb{F}_n}^{\overline{Q}}(N), 1_\Gamma)$. By our assumptions on the sofic approximations, $\lim_{k \rightarrow \infty} \mu_k(U_r) = 0$, hence $\mu(U_r) = 0$. Now let $V_r \subset S_{\mathbb{F}_n}^{\overline{Q}, K}$ be the clopen set of K -colored Schreier graphs G such that the ball $B_r(G, \text{root}(G))$ contains a repetitive path. By our assumptions on the colorings c_k , $\mu_k(V_r) = 0$ for any $k \geq 1$. Hence $\mu(V_r) = 0$. Therefore μ is concentrated on Ω . \square

Now we prove Theorem 2. Observe that we have an F_r -equivariant continuous map $\Sigma : \Omega \rightarrow K^\Gamma$, where \mathbb{F}_n acts on the Bernoulli space K^Γ on the right by $\rho(f)(\gamma) = f(\gamma\kappa(\rho))$ for $\rho \in \mathbb{F}_n, \gamma \in \Gamma$. Then the image of F is a closed Γ -invariant subset in K^Γ , that is a Bernoulli subshift consisting of elements that are given by nonrepetitive K -colorings. The pushforward of μ under Σ is a Γ -invariant probability measure concentrated on Y . By Proposition 3.1, Γ acts freely on Y , hence Theorem 2 follows. \square

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